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Weighted Approximation by Multidimensional Bernstein Operators*

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The purpose of this paper is to give inverse theorems for univariate and multivariate Bernstein operators with some Jacobi weights. A crucial tool in our approach is a decomposition technique which is especially useful for dealing with interpolation theorems. © 1994 Academic Press, Inc.

1. INTRODUCTION

The Bernstein operators on $C[0, 1]$ are given by

$$B_n^*(f, x) = \sum_{k=0}^{n} f(k/n)\, P_{n,k}(x), \tag{1.1}$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

The relation between the rate of convergence and the smoothness of the functions they approximate has been investigated at great length. H. Berens and G. G. Lorentz showed in [3] that for $0 < \alpha = \beta \leqslant 2$,

$$\|(x(1-x))^{-\beta/2}\,(B_n^*(f, x) - f(x))\|_{C[0,1]} = O(n^{-\alpha/2}) \tag{1.2}$$

is equivalent to

$$\|(x(1-x))^{(\alpha-\beta)/2}\,(f(x+h) - 2f(x) + f(x-h))\|_{C[h,1-h]} = O(h^{\alpha}). \tag{1.3}$$

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Z. Ditzian proved the equivalence of (1.2) and (1.3) in the more general cases $0 \leqslant \beta \leqslant \alpha < 2$ and $0 < \alpha < \beta \leqslant 2 - \alpha$ (see [5, 6, 8]). Recently the author solved the left-hand case, $0 < \alpha < \beta < 2$, in [16]. We note that M. Becker, R. A. DeVore, R. J. Nessel, and V. Totik also considered this problem in the direction $\beta \geqslant 0$ (see [1, 2, 13, 14]). The problem in the direction $\beta < 0$, that is, the inverse theorem for weighted approximation, has not been solved although the inverse theorem for weighted approximation by Kantorovich operators was given in 1987 by Z. Ditzian and V. Totik (see [9, Chap. 10]). In this paper we solve this problem. Some ideas in our proof are from [9, 12].

In the multivariate case few inverse results are known. The first result of this kind was given by Z. Ditzian [7] in 1986. Here we discuss the weighted approximation problem for multidimensional Bernstein operators. We deal only with the two dimensional case since the problem in the higher dimensional case can be solved in the same way.

The multidimensional Bernstein operators on the simplex

$$S = \{(x, y) : x, y \geqslant 0, x + y \leqslant 1\} \tag{1.4}$$

are given by

$$B_n(f, x, y) = \sum_{k=0}^{n} \sum_{m=0}^{n-k} f(k/n, m/n)\, P_{n,k,m}(x, y), \tag{1.5}$$

where

$$P_{n,k,m}(x, y) = n!/(k!\, m!(n-k-m)!)\, x^k y^m (1 - x - y)^{n-k-m}.$$

Ditzian's inverse theorem for these operators can be stated as follows.

THEOREM A. *For $f \in C(S)$, $0 < \alpha < 1$, the following statements are equivalent*:

(1) $\|B_n f - f\|_{C(S)} = O(n^{-\alpha})$;

(2)(a) $\|x^\alpha \Delta_{he_1}^2 f(x, y)\|_{L_\infty(x+y \leqslant 3/4, x \geqslant h)} = O(h^{2\alpha})$;

$\|y^\alpha \Delta_{te_2}^2 f(x, y)\|_{L_\infty(x+y \leqslant 3/4, y \geqslant t)} = O(t^{2\alpha})$;

$\|(xy)^{\alpha/2} \Delta_{he_1} \Delta_{te_2} f(x, y)\|_{L_\infty(x+y \leqslant 3/4, x \geqslant h/2, y \geqslant t/2)} = O(h^\alpha t^\alpha)$;

(b) *condition* (a) *is valid for* $f_1(x, y) = f(1 - x - y, y)$;

(c) *condition* (a) *is valid for* $f_2(x, y) = f(x, 1 - x - y)$.

Here $\Delta_{he} f(v) = f(v + he/2) - f(v - he/2)$ *and* $\Delta_{he}^2 f(v) = \Delta_{he}(\Delta_{he} f)(v)$ *for* $e, v \in R^2$ *and* $h \in R_+$. *We also denote* $e_1 = (1, 0)$, $e_2 = (0, 1)$.

In this paper we extend this result to the case of weighted approximation by these multidimensional Bernstein operators with some Jacobi weights. The first main difficulty in this problem is an unbounded property under the usual weight norm, which we discuss in Section 2. To prove our main results we introduce a decomposition technique in Section 6 which is especially crucial for the estimation of the rate of convergence from the smoothness of the function. By our technique a similar weighted approximation problem for multidimensional Bernstein operators on cubes can be easily solved. This technique is also assumed to be valuable for dealing with the weighted approximation in L_p by multidimensional Bernstein-type operators (see [17, 18]).

2. An Unbounded Property

The weights discussed in this paper are the Jacobi weights given by

$$w(x, y) = x^{\beta} y^{\gamma} (1 - x - y)^{\eta}. \tag{2.1}$$

By [9, Chap. 10] we assume that $0 < \beta, \gamma, \eta < 1$.

The inverse theorem for weighted approximation is meant to characterize functions satisfying $\|w(B_n f - f)\|_{C(S)} = O(n^{-\alpha})$ with $0 < \alpha < 1$ by the smoothness of the functions, so a natural norm in $C(S)$ may be defined as

$$\|f\|_w = \|wf\|_{C(S)}.$$

However, with this norm the Bernstein operators are unbounded in $C(S)$, which is a discouraging phenomenon.

Lemma 2.1. *Let $f \in C(S)$. Then we have*

$$\|w(x, y) \sum_{k=1}^{n-1} \sum_{m=1}^{n-k-1} f(k/n, m/n)\, P_{n,k,m}(x, y)\|_{C(S)} \leqslant 4^{\beta+\gamma+\eta+1} \|wf\|_{C(S)}.$$

Proof. Note that [5, Lemma 3.2]

$$\sum_{k=0}^{n} (n/(k+1))^m P_{n,k}(x) \leqslant m!\, x^{-m} \tag{2.2}$$

and

$$\sum_{k=0}^{n} (n/(n-k+1))^m P_{n,k}(x) \leqslant m!(1-x)^{-m}. \tag{2.3}$$

We also observe that

$$P_{n,k,m}(x, y) = P_{n,k}(x)\, P_{n-k,m}(y/(1-x)).$$

Then by using Hölder's inequality we have

$$\begin{aligned}
&\left| x^{\beta} y^{\gamma} (1-x-y)^{\eta} \sum_{k=1}^{n-1} \sum_{m=1}^{n-k-1} f(k/n, m/n)\, P_{n,k,m}(x, y) \right| \\
&\leqslant \|wf\|_{\infty}\, x^{\beta} \sum_{k=1}^{n-1} P_{n,k}(x)\, y^{\gamma}(1-x-y)^{\eta}\, (k/n)^{-\beta}\, ((n-k)/n)^{-\gamma-\eta} \\
&\quad \times \left\{ 2^{\gamma+\eta} \sum_{m=1}^{n-k} (((n-k)/(m+1))^{\gamma} \right. \\
&\quad \left. + ((n-k)/(n-k-m+1))^{\eta})\, P_{n-k,m}(y/(1-x)) \right\} \\
&\leqslant \|wf\|_{\infty}\, x^{\beta} \sum_{k=1}^{n-1} P_{n,k}(x)\, y^{\gamma}(1-x-y)^{\eta}\, (k/n)^{-\beta}\, ((n-k)/n)^{-\gamma-\eta} \\
&\quad \times 2^{\gamma+\eta} \{((1-x)/y)^{\gamma} + ((1-x)/(1-x-y))^{\eta}\} \\
&\leqslant 2^{\gamma+\eta+1}\, \|wf\|_{\infty}\, x^{\beta}(1-x)^{\gamma+\eta}\, 2^{\gamma+\eta+\beta} \\
&\quad \times \sum_{k=1}^{n-1} P_{n,k}(x) \{(n/(k+1))^{\beta} + (n/(n-k+1))^{\gamma+\eta}\} \\
&\leqslant 2^{2+2\gamma+2\eta+\beta}\, \|wf\|_{C(S)}.
\end{aligned}$$

PROPOSITION 2.2. *For any $n \in N$ the Bernstein operator B_n is unbounded on $(C(S), \|\cdot\|_w)$.*

Proof. Let $f_p(x, y) = (1-x)\, y(1-x-y)/(1/p + x^{\beta}) \in C(S)$ for $p \in N$. We have

$$\|wf_p\|_{C(S)} \leqslant 1.$$

On the other hand, by Lemma 2.1 we have

$$\begin{aligned}
\|wB_n(f_p)\|_{C(S)} &\geqslant \left\| w \sum_{m=1}^{n-1} P_{n,0,m} f_p(0, m/n) \right\|_{C(S)} - 4^{\beta+\gamma+\eta+1}\, \|wf_p\|_{C(S)} \\
&\geqslant p \left\| w(x, y) \sum_{m=1}^{n-1} P_{n,0,m}(x, y)\, m(n-m)\, n^{-2} \right\|_{C(S)} - 4^{\beta+\gamma+\eta+1}.
\end{aligned}$$

Therefore, letting $p \to \infty$, we know that Proposition 2.2 holds.

Thus, it is natural for us to discuss the weighted approximation problem only in the space $C_0(S)$, where $C_0(S)=\{f\in C(S): f$ vanishes in the boundary of $S\}$. With this restriction we have the bounded property.

PROPOSITION 2.3. *Let $f\in C_0(S)$, $n\in N$. Then we have*

$$\|B_n f\|_w \leqslant 4^{1+\beta+\gamma+\eta}\|f\|_w.$$

If we discuss the weighted approximation problem for multidimensional integral-type operators in L_p $(1\leqslant p\leqslant\infty)$ [17, 18], we need not make such a restriction since multidimensional Bernstein–Durrmeyer operators and Kantorovich operators are bounded in the weight norm.

3. Main Results

In this section we state our main results in the univariate and multivariate cases. We fix some constants $\frac{2}{3}<a=\frac{33}{48}<b=\frac{17}{24}<c<\frac{3}{4}$. We also denote M_s as a constant depending only on s, which may be different at each occurrence.

To measure the smoothness of the functions we use the differences defined as $\Delta_{he}f(v)=f(v+he)-f(v)$ and $\Delta_{he}^2 f(v)=\Delta_{he}(\Delta_{he}f)(v)$ for vectors $v, e\in R^2$ or R and $h\in R_+$. We give an equivalence between the rate of convergence and the smoothness of the functions by using the technique of interpolation spaces, which has been developed by many mathematicians (see [9, 10, 13, 15]). So we give some notations first.

Let $w_1(x,y)=w(1-x-y,y)$, $w_2(x,y)=w(x,1-x-y)$.

We define the Peetre K-functional on $C_0(S)$ as

$$K(f,t)_w=\inf_{g\in D}\{\|w(f-g)\|_\infty+t\phi(g)\}\qquad(t>0), \tag{3.1}$$

where D is the weighted Sobolev space defined by

$$D=\left\{g\in C_0(S): g, \frac{\partial}{\partial x}g, \frac{\partial}{\partial y}g\in AC_{\mathrm{loc}}, \phi(g)<\infty\right\} \tag{3.2}$$

and

$$\phi(g)=\|wg\|_\infty+\phi_w(g)+\phi_{w_1}(g_1)+\phi_{w_2}(g_2), \tag{3.3}$$

$$\begin{aligned}\phi_w(g)=\max\Bigg\{&\left\|w(x,y)\,x\left(\frac{\partial^2}{\partial x^2}g\right)(x,y)\right\|_{L_\infty(x+y\leqslant 3/4)},\\ &\left\|w(x,y)\,y\left(\frac{\partial^2}{\partial y^2}g\right)(x,y)\right\|_{L_\infty(x+y\leqslant 3/4)},\\ &\left\|w(x,y)\sqrt{xy}\left(\frac{\partial^2}{\partial x\,\partial y}g\right)(x,y)\right\|_{L_\infty(x+y\leqslant 3/4)}\Bigg\}.\end{aligned} \tag{3.4}$$

With all the above expressions we can now give the following characterization of the rate of convergence for multivariate operators with Jacobi weights by the smoothness of the function.

THEOREM 1. *Let* $f \in C_0(S)$, $0<\alpha<1$, *and* $B_n(f, x, y)$ *be given by* (1.5). *Then the following statements are equivalent*:

(1) $\|w(B_n f - f)\|_{C(S)} = O(n^{-\alpha})$; (3.5)

(2) $K(f, t)_w = O(t^\alpha)$; (3.6)

(3)(a) $\|w(x, y)\, x^\alpha\, \Delta_{he_1}^2 f(x, y)\|_{L_\infty(x+y\leqslant c)} = O(h^{2\alpha})$; (3.7)

$\|w(x, y)\, y^\alpha\, \Delta_{te_2}^2 f(x, y)\|_{L_\infty(x+y\leqslant c)} = O(t^{2\alpha})$; (3.8)

$\|w(x, y)(xy)^{\alpha/2}\, \Delta_{he_1} \Delta_{te_2} f(x, y)\|_{L_\infty(x+y\leqslant c)} = O(h^\alpha t^\alpha)$; (3.9)

(b) *condition* (a) *is valid for* f_1 *and* w_1;

(c) *condition* (a) *is valid for* f_2 *and* w_2.

Remark. In view of Ditzian's Theorem A, Theorem 1 is natural. We mention that the saturation case $\alpha = 1$ remains open even for $w = 1$. Similar results hold for multidimensional operators on cubes and the proofs are simpler, so we omit them here.

In the univariate case we also have the following characterization theorem.

THEOREM 2. *Let* $f \in C[0, 1]$, $0<\alpha<1$, $0<\beta$, $\gamma<1$, *and* $B_n^*(f, x)$ *be given by* (1.1). *Then*

$$\|x^\beta(1-x)^\gamma\,(B_n^*(f, x) - f(x))\|_{C[0,1]} = O(n^{-\alpha}) \tag{3.10}$$

if and only if

$$\|x^\beta(1-x+2h)^\gamma\,\Delta_h^2 f(x)\|_{L_\infty[0\leqslant x\leqslant 1-2h]} = O(h^{2\alpha}). \tag{3.11}$$

Remark. In the weighted approximation with $\beta, \gamma > 0$, the saturation case has remained open for Bernstein operators as well as for integral-type operators.

We only illustrate our method and prove Theorem 1, since the proof of Theorem 2 is simpler, as is clear from the discussion in Section 5. In Section 4 we show that (3.5) implies (3.6). In Section 5 we give a direct estimate in the univariate case. Then, in Section 6 we prove the final implications of Theorem 1 by our decomposition technique for multidimensional Bernstein operators.

4. Bernstein-Type Inequalities

To prove that (3.5) implies (3.6) we use the standard method, as in [9, 10, 13]. It is sufficient to prove the Bernstein-type inequalities

$$\phi(B_n f) \leqslant M\, n\, \|f\|_w, \qquad \text{for} \quad f \in C_0(S)$$

and

$$\phi(B_n f) \leqslant M \phi(f), \qquad \text{for} \quad f \in D.$$

To obtain these inequalities the following preliminary results are necessary.

Lemma 4.1. *For $B_n(f(s, t), x, y)$ given by (1.5), we have*

$$B_n(1, x, y) = 1; \tag{4.1}$$

$$B_n(s, x, y) = x; \tag{4.2}$$

$$B_n((s-x)^2, x, y) = x(1-x)/n; \tag{4.3}$$

$$B_n(st, x, y) = xy(n-1)/n. \tag{4.4}$$

Lemma 4.2 [7, Lemma 2.1]. *Let $f \in C_0(S)$. Then we have*

$$\left(\frac{\partial}{\partial x} B_n f\right)(x, y) = (x(1-x-y))^{-1} \sum_{k=0}^{n} \sum_{m=0}^{n-k} f(k/n, m/n)\, P_{n,k,m}(x, y) \times (k(1-x-y) - (n-k-m)x) \tag{4.5}$$

$$= \sum_{k=1}^{n} \sum_{m=0}^{n-k} P_{n-1,k-1,m}(x, y)(f(k/n, m/n) - f((k-1)/n, m/n)); \tag{4.6}$$

$$\left(\frac{\partial^2}{\partial x^2} B_n f\right)(x, y) = x^{-2}(1-x-y)^{-2} \sum_{k=0}^{n} \sum_{m=0}^{n-k} f(k/n, m/n) \times P_{n,k,m}(x, y)(k(k-1)(1-x-y)^2 - 2k(n-k-m) \times x(1-x-y) + x^2(n-k-m)(n-k-m-1)) \tag{4.7}$$

$$= n(n-1) \sum_{k=2}^{n} \sum_{m=0}^{n-k} P_{n-2,k-2,m}(x, y)(f(k/n, m/n) - 2f((k-1)/n, m/n) + f((k-2)/n, m/n)); \tag{4.8}$$

$$\left(\frac{\partial^2}{\partial x\,\partial y}B_nf\right)(x,y)=(xy)^{-1}(1-x-y)^{-2}\sum_{k=0}^{n}\sum_{m=0}^{n-k}f(k/n,m/n)\,P_{n,k,m}(x,y)$$
$$\times(km(1-x-y)^2-(ky+mx)(n-k-m)(1-x-y)$$
$$+(n-k-m)(n-k-m-1)\,xy) \tag{4.9}$$
$$=n(n-1)\sum_{k=1}^{n}\sum_{m=1}^{n-k}P_{n-2,k-1,m-1}(x,y)(f(k/n,m/n)$$
$$-f((k-1)/n,m/n)-f(k/n,(m-1)/n)$$
$$+f((k-1)/n,(m-1)/n)). \tag{4.10}$$

Remark. Expressions for the second variable in the above lemmas can be given in the same way.

Now we can prove the first Bernstein-type inequality.

LEMMA 4.3. *Let $f\in C_0(S)$. Then we have*

$$\phi(B_nf)\leqslant M\,n\,\|f\|_w, \tag{4.11}$$

where the constant M is independent of f and n.

Proof of Lemma 4.3. It is sufficient to prove

$$\left\|w(x,y)\,x\left(\frac{\partial^2}{\partial x^2}B_nf\right)(x,y)\right\|_{L_\infty(x+y\leqslant 3/4)}\leqslant M\,n\,\|f\|_w \tag{4.12}$$

and

$$\left\|w(x,y)\sqrt{xy}\left(\frac{\partial^2}{\partial x\,\partial y}B_nf\right)(x,y)\right\|_{L_\infty(x+y\leqslant 3/4)}\leqslant M\,n\,\|f\|_w. \tag{4.13}$$

We first prove (4.12). Denote $Q=P_{n,k,m}(x,y)((k(1-x-y)-(n-k-m)x)^2+k(1-x-y)^2+(n-k-m)\,x^2)$. We note that $0<\beta,\ \gamma,\ \eta<1$. Let $x+y\leqslant\frac{3}{4}$. Then by Hölder's inequality and (4.7) we have

$$\left|\left(\frac{\partial^2}{\partial x^2}B_nf\right)(x,y)\right|$$
$$\leqslant x^{-2}(1-x-y)^{-2}\,\|f\|_w\left(\sum(k/n)^{-3}\,Q\right)^{\beta/3}\left(\sum(m/n)^{-3}\,Q\right)^{\gamma/3}$$
$$\times\left(\sum((n-k-m)/n)^{-3}\,Q\right)^{\eta/3}\left(\sum Q\right)^{1-(\beta+\gamma+\eta)/3}. \tag{4.14}$$

Here all the sums are taken for $k,m\geqslant 1$ and $k+m<n$.

Therefore, we need to estimate the sums in (4.14). For $x+y\leqslant\frac{3}{4}$ we can obtain

$$\begin{aligned}\sum_{k=1}^{n-1}\sum_{m=1}^{n-k-1}(k/n)^{-3}Q &\leqslant 4^3x^{-3}\sum_{k=1}^{n-1}\sum_{m=1}^{n-k-1}P_{n+3,k+3,m}(x,y)\\ &\quad\times\{((k+3)(1-x-y)-((n+3)-(k+3)-m)x)^2\\ &\quad+16k(1-x-y)^2+7(n-k-m)x\}\\ &\leqslant 4^3x^{-3}((n+3)\,x(1-y)+23(n+3)\,x(1-x-y))\\ &\leqslant 4^6(n+3)\,x^{-2};\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n-1}\sum_{m=1}^{n-k-1}Q &\leqslant B_n(n^2(s(1-x-y)-(1-s-t)x)^2\\ &\quad+ns(1-x-y)^2+n(1-s-t)\,x^2,x,y)\\ &\leqslant 16n\,x.\end{aligned}$$

The other two sums in (4.14) can be estimated in the same way and we obtain for $x+y\leqslant\frac{3}{4}$

$$\begin{aligned}&\left|w(x,y)\,x\left(\frac{\partial^2}{\partial x^2}B_nf\right)(x,y)\right|\\ &\leqslant M_{\beta,\gamma,\eta}x^\beta y^\gamma(1-x-y)^\eta\,xx^{-2}(1-x-y)^{-2}\,\|f\|_w\\ &\quad\times(nx^{-2})^{\beta/3}\,(nxy^{-3})^{\gamma/3}\,(nx(1-x-y)^{-3})^{\eta/3}\,(nx)^{1-(\beta+\gamma+\eta)/3}\\ &\leqslant M\,n\,\|f\|_w.\end{aligned}$$

Hence, (4.12) holds.

Note that

$$\begin{aligned}&|(xy)^{-1}\,\{km(1-x-y)^2-(ky+mx)(n-k-m)\\ &\quad\times(1-x-y)+(n-k-m)^2\,xy\}|\\ &\leqslant x^{-2}(k(1-x-y)-(n-k-m)x)^2\\ &\quad+y^{-2}(m(1-x-y)-(n-k-m)\,y)^2.\end{aligned}\tag{4.15}$$

The proof of (4.13) can then be easily obtained by Hölder's inequality, and the same estimates as those for (4.12) obtained for the two variables. The proof of Lemma 4.3 is now complete.

Next, we give the second inequality. To this end, we need another norm in D defined as

$$\phi^*(f)=\|f\|_w+\phi_w^*(f)+\phi_{w_1}^*(f_1)+\phi_{w_2}^*(f_2),\tag{4.16}$$

where

$$\phi_w^*(f) = \max\left\{ \left\| w(x,y)\, x \left(\frac{\partial^2}{\partial x^2} f\right)(x,y) \right\|_{L_\infty(x+y\leqslant 2/3)}, \right.$$
$$\left\| w(x,y)\sqrt{xy} \left(\frac{\partial^2}{\partial x\,\partial y} f\right)(x,y) \right\|_{L_\infty(x+y\leqslant 2/3)},$$
$$\left. \left\| w(x,y)\, y \left(\frac{\partial^2}{\partial y^2} f\right)(x,y) \right\|_{L_\infty(x+y\leqslant 2/3)} \right\}. \tag{4.17}$$

This norm is crucial in the proof of our second inequality, but it is equivalent to the previous norm ϕ in D, which we show as follows.

LEMMA 4.4. *For the two norms ϕ and ϕ^* given by* (3.3) *and* (4.16) *in D, we have*

$$\phi(f) \leqslant C\phi^*(f), \tag{4.18}$$

where C is a constant independent of $f \in D$.

The proof of Lemma 4.4 is easy if we write out the expressions of $\phi_{w_1}^*(f_1)$ and $\phi_{w_2}^*(f_2)$ explicitly.

Our second inequality can then be stated as follows.

LEMMA 4.5. *Let $f \in D$. Then we have*

$$\phi(B_n f) \leqslant L\phi(f), \tag{4.19}$$

where L is independent of f and n.

Proof of Lemma 4.5. By Lemma 4.4 we need only prove that

$$\phi_w^*(B_n f) \leqslant L\phi(f).$$

Let us first assume that $f|_{x+y\geqslant b} = 0$. Under this assumption we prove that

$$\phi_w^*(B_n f) \leqslant L\phi_w(f). \tag{4.20}$$

By (4.8) we have for $x + y \leqslant \frac{2}{3}$

$$\left| w(x,y)\, x \left(\frac{\partial^2}{\partial x^2} B_n f\right)(x,y) \right|$$
$$= \left| n(n-1)\, w(x,y) x \sum_{k=2}^{n} \sum_{m=1}^{n-k} P_{n-2,k-2,m}(x,y) \right.$$
$$\left. \times \int_0^{1/n} \int_0^{1/n} \left(\frac{\partial^2}{\partial x^2} f\right)((k-2)/n + u + v, m/n)\, du\, dv \right|$$

$$\leqslant n^2 w(x, y) x \sum_{k=2}^{3n/4} \sum_{m=1}^{3n/4-k} P_{n-2,k-2,m}(x, y)$$

$$\times \int_0^{1/n} \int_0^{1/n} \phi_w(f)((k-2)/n+u+v)^{-\beta-1} (m/n)^{-\gamma} 4^\eta \, du \, dv$$

$$\leqslant M_\beta 4^{\eta+2\beta+1} n^2 x^{\beta+1} y^\gamma \phi_w(f) \sum_{k=2}^{3n/4} \sum_{m=1}^{3n/4-k} P_{n-2,k-2,m}(x, y)$$

$$\times (n/m)^\gamma (n/(k-1))^{\beta+1} n^{-2}$$

$$\leqslant M_{\beta,\gamma,\eta} x^{\beta+1} y^\gamma \phi_w(f) x^{-\beta-1} y^{-\gamma}$$

$$\leqslant L\phi_w(f).$$

Here L depends only on β, γ, and η. In the above estimates we have used (2.2), Hölder's inequality, and the inequality (in [2])

$$\int_{-h/2}^{h/2} \int_{-h/2}^{h/2} (x+s+t)^{-\beta'} (1-x-s-t)^{-2\beta'} \, ds \, dt \leqslant M_{\beta'} h^2 x^{-\beta'} (1-x)^{-2\beta'}, \tag{4.21}$$

where $x \in [h, 1-h]$, $0 < h \leqslant \frac{1}{8}$, and $0 < \beta' = (\beta+1)/2 < 1$.

The other two terms in the definition of $\phi_w^*(B_n f)$ can be estimated in the same way and we have proved (4.20) under the assumption that $f|_{x+y \geqslant b} = 0$.

To complete the proof for any $f \in D$ we choose a function $\psi \in C^\infty(S)$ such that

$$\psi|_{x+y \leqslant 33/48} = 1$$

and

$$\psi|_{x+y \geqslant 17/24} = 0.$$

Then, for any $f \in D$ we have $\psi f \in D$ and by (4.20),

$$\phi_w^*(B_n f) \leqslant \phi_w^*(B_n(f - \psi f)) + L\phi_w(\psi f).$$

We observe that $(f-\psi f)|_{x+y \leqslant 33/48} = 0$ and $|(f-\psi f)(k/n, m/n)| \leqslant (1 + \|\psi\|_\infty) n^{\beta+\gamma+\eta} \|f\|_w$. By (4.7), (4.9), and Lemma 4.6 below, we have

$$\phi_w^*(B_n(f-\psi f)) \leqslant L_1 \|f\|_w$$

with the constant L_1 independent of f and n.

For the second term we have

$$\phi_w(\psi f) \leqslant \Big(\|\psi\|_\infty + \Big\|\frac{\partial}{\partial x}\psi\Big\|_\infty + \Big\|\frac{\partial}{\partial y}\psi\Big\|_\infty + \Big\|\frac{\partial^2}{\partial x^2}\psi\Big\|_\infty + \Big\|\frac{\partial^2}{\partial x\,\partial y}\psi\Big\|_\infty + \Big\|\frac{\partial^2}{\partial y^2}\psi\Big\|_\infty \Big) \times \Big(\phi_w(f) + \|f\|_w + \Big\| w(x, y)\Big(\frac{\partial}{\partial x} f\Big)(x, y)\Big\|_{L_\infty(x+y\leqslant 3/4)} + \Big\| w(x, y)\Big(\frac{\partial}{\partial y} f\Big)(x, y)\Big\|_{L_\infty(x+y\leqslant 3/4)} \Big).$$

Then, Lemma 4.7 below, yields

$$\phi_w(\psi f) \leqslant M_\psi \phi(f).$$

The proof of Lemma 4.5 is now complete.

LEMMA 4.6. *There exists an absolute constant M_0 such that*

$$P_{n,k,m}(x, y) \leqslant M_0 n^{-6} \tag{4.22}$$

for $x + y \leqslant \frac{2}{3}$ and $(k+m)/n \geqslant \frac{33}{48}$.

The proof of Lemma 4.6 is easy and we omit it here.

LEMMA 4.7 [9, Lemma 10.5.1]. *Let $w^*(x) = x^a(1-x)^b$ with $a, b \in (0, 1)$, and*

$$D^* = \{ g \in C[0, 1] : g' \in AC_{\text{loc}},\ w^*(x)\, x(1-x)\, g''(x) \in L_\infty[0, 1] \}.$$

Then we have a constant C independent of $g \in D^*$ such that

$$\|w^* g'\|_\infty \leqslant C(\|w^* g\|_\infty + \|w^*(x)\, x(1-x)\, g''(x)\|_\infty).$$

With all the above lemmas we have proved that (3.5) implies (3.6) in Theorem 1. This first step can be easily extended to the weighted approximation in L_p by the multidimensional operators given in [17, 18].

In what follows we give the other steps in the proof of Theorem 1. First, we give a direct theorem for univariate operators.

5. A Direct Theorem for Univariate Operators

In this section we give a direct theorem for weighted approximation by univariate Bernstein operators. Then, by means of this theorem we prove Theorem 1 in the next section, by our decomposition technique.

We first give the definition of the so-called weighted Lipschitz constants for the functions in $C[0, 1]$. Then we state and prove our direct theorem by means of these constants.

Definition. Let $0 \leqslant \theta < 1$, $0 < \alpha < 1$. Then the weighted Lipschitz constants for $g \in C[0, 1]$ are defined by

$$L(g) = \sup_{\substack{0 \leqslant h \leqslant (c-b)/2 \\ x}} \{|h^{-2\alpha} x^{\theta+\alpha} \Delta_h^2 g(x)|\} \tag{5.1}$$

and

$$L'(g) = \sup_{0 < h \leqslant (c-b)/2} \{|h^{\theta-\alpha} \Delta_h^2 g(0)|\}. \tag{5.2}$$

Concerning these two constants we have the following relation.

Lemma 5.1. *Suppose that $L(g) < \infty$ for $g \in C[0, 1]$. Then we have*

$$L'(g) \leqslant M_{\alpha,\theta}(L(g) + \|g\|_\infty). \tag{5.3}$$

Proof of Lemma 5.1. If $\theta \geqslant \alpha$, then we have

$$L'(g) \leqslant 4\, \|g\|_\infty.$$

If $\theta < \alpha$, we introduce a local modulus of smoothness at zero as

$$v(t) = \sup_{0 < h \leqslant t} |\Delta_h^2 g(0)|.$$

For any $h \in [0, t]$, we choose $d = h/2$ and use the Steklov function in [12, p. 194] defined by

$$g_d(s) = (2/d)^2 \int_0^{d/2} \int_0^{d/2} (2g(s+u+v) - g(s+2u+2v))\, du\, dv. \tag{5.4}$$

Then, we have

$$\begin{aligned}
|\Delta_h^2 g_d(0)| &= \left| \int_0^h \int_0^h g_d''(u+v)\, du\, dv \right| \\
&\leqslant \int_0^h \int_0^h |d^{-2}(8\Delta_{d/2}^2 g(u+v) - \Delta_d^2 g(u+v))|\, du\, dv \\
&\leqslant 9d^{-2} \int_0^h \int_0^h d^{2\alpha}(u+v)^{-\alpha-\theta}\, du\, dv\, L(g) \\
&\leqslant M'_{\alpha,\theta}\, d^{2\alpha-2} h^{2-\alpha-\theta} L(g)
\end{aligned}$$

and

$$|g_d(0)-g(0)| \leqslant v(h/2).$$

Thus,

$$\begin{aligned}|\Delta_h^2 g(0)| &\leqslant |\Delta_h^2 g_d(0)| + |g_d(0)-g(0)| + |g_d(2h)-g(2h)| + 2\,|g_d(h)-g(h)| \\ &\leqslant 4M'_{\alpha,\theta} h^{\alpha-\theta} L(g) + v(h/2) + 3d^{2\alpha} h^{-\theta-\alpha} L(g) \\ &\leqslant v(t/2) + (4M'_{\alpha,\theta}+3)\, t^{\alpha-\theta} L(g).\end{aligned}$$

Hence, by induction,

$$\begin{aligned}v(t) &\leqslant v(t/2) + (4M'_{\alpha,\theta}+3)\, t^{\alpha-\theta} L(g) \\ &\leqslant \cdots \\ &\leqslant (4M'_{\alpha,\theta}+3)\, t^{\alpha-\theta}/(1-2^{\theta-\alpha}).\end{aligned}$$

Therefore, we have

$$L'(g) \leqslant \sup_{0<t\leqslant (c-b)/2} \{t^{\theta-\alpha} v(t)\} \leqslant (4M'_{\alpha,\theta}+3)\, L(g)/(1-2^{\theta-\alpha}).$$

The proof of Lemma 5.1 is complete.

With the above preliminary result, we can now give our direct theorem as follows.

LEMMA 5.2. *Suppose that $g \in C[0,1]$ satisfies $g|_{x \geqslant b} = 0$ and $L(g) < \infty$. Then we have*

$$\|x^\theta (B_n^*(g,x) - g(x))\|_{L_\infty(x\leqslant 2/3)} \leqslant M_{\alpha,\theta} n^{-\alpha} (L(g) + \|g\|_\infty). \qquad (5.5)$$

Proof of Lemma 5.2. By Lemma 5.1 we need only prove (5.5) with the term $L(g)+\|g\|_\infty$ replaced by $L(g)+L'(g)$.

For $x \in (0, \frac{2}{3}]$, let $d = \sqrt{x/n} \leqslant \sqrt{6}/3$. We use the Steklov function g_d defined by (5.4). By the computations in [12, p. 294] we have

$$\begin{aligned}|g(t)-g_d(t)| &\leqslant (2/d)^2 \int_0^{d/2}\int_0^{d/2} (u+v)^{2\alpha}\, t^{-\theta-\alpha}\, du\, dv\, L(g) \\ &\leqslant t^{-\theta-\alpha}\, d^{2\alpha} L(g) \qquad \text{for} \quad t>0; \\ |g_d''(t)| &\leqslant 9t^{-\theta-\alpha}\, d^{2\alpha-2} L(g) \qquad \text{for} \quad t>0; \\ |g(0)-g_d(0)| &\leqslant (2/d)^2 \int_0^{d/2}\int_0^{d/2} (u+v)^{\alpha-\theta}\, du\, dv\, L'(g) \\ &\leqslant M_{\alpha,\theta}\, d^{\alpha-\theta} L'(g).\end{aligned}$$

To estimate (5.5) we need a class of modified Bernstein operators given by

$$B_n^{**}(f, x) = \sum_{k=1}^{n} f(k/n)\, P_{n,k}(x).$$

Then, we have for $x \in (0, \frac{2}{3}]$

$$\begin{aligned} x^\theta\,|B_n^*(g, x) - g(x)| &\leqslant x^\theta\,|g_d(x) - g(x)| + x^\theta B_n^{**}(|g_d - g|, x) \\ &\quad + x^\theta P_{n,0}(x)\,|g(0) - g_d(0)| + x^\theta\,|B_n^*(g_d, x) - g_d(x)| \\ &:= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

By the above estimates for g_d we obtain

$$\begin{aligned} J_1 &\leqslant x^\theta x^{-\theta-\alpha}(x/n)^\alpha\, L(g) = n^{-\alpha} L(g); \\ J_2 &\leqslant x^\theta (B_n^{**}(t^{-2}, x))^{(\theta+\alpha)/2}\, d^{2\alpha} L(g) \leqslant 6n^{-\alpha} L(g); \\ J_3 &\leqslant x^\theta (1-x)^n\, M_{\alpha,\theta}(x/n)^{(\alpha-\theta)/2} L'(g) \\ &\leqslant M_{\alpha,\theta}(x(1-x)^{n-1})^{(\alpha+\theta)/2}\, n^{(\theta-\alpha)/2} L'(g) \leqslant M_{\alpha,\theta} n^{-\alpha} L'(g). \end{aligned}$$

The main difficulty exists in the estimates of J_4. We discuss two cases.

If $\alpha + \theta \leqslant 1$, then $(t-u)\,u^{-\theta-\alpha}$ is monotone for $u \in (x, t)$ or (t, x). Hence,

$$\begin{aligned} J_4 &\leqslant x^\theta B_n^*\left(\left|\int_x^t |(t-u)\, g_d''(u)|\, du\right|, x\right) \\ &\leqslant 9x^\theta B_n^*((t-x)^2\, x^{-\theta-\alpha}\, d^{2\alpha-2} L(g), x) \\ &\leqslant 9n^{-\alpha} L(g). \end{aligned}$$

The second case is $\alpha + \theta > 1$. In this case we use a different method to estimate J_4.

We observe that

$$\begin{aligned} J_4 &\leqslant x^\theta B_n^*\left(\left|\int_x^t (t-u)\, g_d''(u)\, du\right|, x\right) \\ &\leqslant 9x^\theta\, d^{2\alpha-2} L(g)\, B_n^*\left(\frac{t-x}{x}\int_x^t u^{1-\theta-\alpha}\, du, x\right). \end{aligned}$$

Therefore, using Hölder's inequality we have for $x \in (0, 1/n]$

$$\begin{aligned} J_4 &\leqslant 9x^{\theta-1}\, d^{2\alpha-2} L(g)\, B_n^*(|t-x|^{3-\theta-\alpha}, x)/(2-\theta-\alpha) \\ &\leqslant 9x^{\theta-1}\, d^{2\alpha-2} L(g)(x(1-x)/n)^{(3-\theta-\alpha)/2}/(2-\theta-\alpha) \\ &\leqslant 9n^{-\alpha} L(g)/(2-\theta-\alpha). \end{aligned}$$

For $x \in (1/n, \frac{2}{3}]$ we also have

$$J_4 \leqslant 9x^{\theta-1} d^{2\alpha-2} L(g)\{B_n^{**}((t-x)^2 (t^{1-\theta-\alpha} + x^{1-\theta-\alpha}), x) + P_{n,0}(x)\, x^{3-\theta-\alpha}/(2-\theta-\alpha)\}$$

$$\leqslant 36 n^{-\alpha} L(g)/(2-\theta-\alpha) + 9x^{\theta-1} d^{2\alpha-2} L(g)(B_n^{**}((t-x)^4, x))^{1/2} \times (B_n^{**}(t^{-2}, x))^{(\theta+\alpha-1)/2}$$

$$\leqslant M_{\alpha,\theta} n^{-\alpha} L(g).$$

Thus, combining all the above estimates we have

$$\|x^\theta (B_n^*(g, x) - g(x))\|_{L_\infty(x \leqslant 2/3)} \leqslant M_{\alpha,\theta} n^{-\alpha}(L(g) + L'(g)).$$

Hence, Lemma 5.2 holds.

By our method here we can give characterization theorems for weighted approximation by other exponential-type operators. It is also possible to give other direct results, not only in the non-optimal Lipschitz case. A crucial tool in these approaches is a certain Steklov-type function as given by (5.4). For the operators defined on infinite intervals it is interesting to consider weighted approximation with weights other than polynomial weights.

6. Proof of the Main Results

For the multidimensional Bernstein operators given by (1.5), our decomposition technique can be expressed as

$$B_n(f, x, y) = \sum_{k=0}^{n-1} P_{n,k}(x)\{B_{n-k}^*(f_{k/n}, y/(1-x)) - f_{k/n}(y/(1-x))\} + \{B_n^*(f^{y/(1-x)}, x) - f^{y/(1-x)}(x)\}. \tag{6.1}$$

Here, for $s, t \in [0, 1]$ we denote f_s and $f^t \in C[0, 1]$ as

$$f_s(t) = f^t(s) = f(s, (1-s)t). \tag{6.2}$$

This decomposition technique for multidimensional Bernstein operators is crucial in the proof of our main results. Similar techniques can be used for other multidimensional operators on different domains (see [17, 18]).

We have now decomposed the multidimensional Bernstein operators into univariate Bernstein operators. Therefore, we can use our direct result in Section 5 for univariate operators to prove Theorem 1.

Proof of Theorem 1. Suppose that (3.6) holds. We show that the conditions in (3) are satisfied. We only prove (3.7) since the other conditions can be proved in the same way.

Let $x+y\leqslant c$, $h<\frac{3}{8}-c/2$. Then, for any $g\in D$ we have

$$\begin{aligned}|w(x,y)\,x^\alpha\,\Delta^2_{he_1}f(x,y)| &\leqslant 4w(x,y)\,x^\alpha\,\|f-g\|_w\,x^{-\beta}y^{-\gamma}(1-x-y-2h)^{-\eta}\\ &\quad+w(x,y)\,x^\alpha\int_0^h\int_0^h\left|\left(\frac{\partial^2}{\partial x^2}g\right)(x+u+v,y)\right|du\,dv\\ &\leqslant 4^{1+\eta}x^\alpha\,\|f-g\|_w+4^\eta x^{\alpha-1}h^2\phi(g)\\ &\leqslant 4^{1+\eta}x^\alpha\{\|f-g\|_w+h^2x^{-1}\phi(g)\}.\end{aligned}$$

Hence,

$$\begin{aligned}|w(x,y)\,x^\alpha\,\Delta^2_{he_1}f(x,y)| &\leqslant 4^{1+\eta}x^\alpha K(f,h^2/x)_w\\ &\leqslant 4^{1+\eta}x^\alpha M(h^2/x)^\alpha\\ &\leqslant 4^{1+\eta}Mh^{2\alpha}.\end{aligned}$$

Therefore, (3.7) holds. We have proved the implication "(2) ⇒ (3)".

Now we prove the final implication. Suppose that the conditions in (3) are satisfied and the terms in (3.7)–(3.9) are bounded by $Mh^{2\alpha}$, $Mt^{2\alpha}$, and $Mh^\alpha t^\alpha$, respectively. To complete our proof, it is sufficient to prove

$$\|w(x,y)(B_n(f,x,y)-f(x,y))\|_{L_\infty(x+y\leqslant 2/3)}=O(n^{-\alpha}). \tag{6.3}$$

By Lemma 4.6 we can assume that $f|_{x+y\geqslant b}=0$.

Note that f vanishes in the boundary of S. By our decomposition formula (6.1) and Lemma 5.2 we have for $x+y\leqslant\frac{2}{3}$ and $x,y>0$

$$\begin{aligned}&|w(x,y)(B_n(f,x,y)-f(x,y))|\\ &\leqslant x^\beta\sum_{k=1}^{[bn]}P_{n,k}(x)\,|(y/(1-x))^\gamma\\ &\quad\times\{B^*_{n-k}(f_{k/n},y/(1-x))-f_{k/n}(y/(1-x))\}|\\ &\quad+y^\gamma\,|x^\beta\{B^*_n(f^{y/(1-x)},x)-f^{y/(1-x)}(x)\}|\\ &\leqslant x^\beta\sum_{k=1}^{[bn]}P_{n,k}(x)\,\|z^\gamma(B^*_{n-k}(f_{k/n},z)-f_{k/n}(z))\|_{L_\infty(z\leqslant 2/3)}\\ &\quad+y^\gamma\,\|z^\beta(B^*_n(f^{y/(1-x)},z)-f^{y/(1-x)}(z))\|_{L_\infty(z\leqslant 2/3)}\\ &\leqslant x^\beta\sum_{k=1}^{[bn]}P_{n,k}(x)\,M_{\alpha,\gamma}(n-k)^{-\alpha}\,(L(f_{k/n})+\|f\|_\infty)\\ &\quad+y^\gamma M_{\alpha,\beta}n^{-\alpha}\,(L(f^{y/(1-x)})+\|f\|_\infty).\end{aligned} \tag{6.4}$$

Then, we need only compute the weighted Lipschitz constant of the univariate functions. We note that $f_{k/n}|_{(b,1]}=0$, $f^z|_{(b,1]}=0$.

We first compute $L(f_{k/n})$. Let $0<x\leqslant b$, $0<h\leqslant (c-b)/2$, $\theta=\gamma$, $0<k\leqslant bn$. Then, we have

$$\begin{aligned}
&|h^{-2\alpha}x^{\gamma+\alpha}\Delta_h^2 f_{k/n}(x)| \\
&\quad = h^{-2\alpha}x^{\gamma+\alpha}\,|\Delta^2_{(1-k/n)he_2} f(k/n, (1-k/n)x)| \\
&\quad \leqslant 4^{2\eta}h^{-2\alpha}x^{\gamma+\alpha}M((1-k/n)h)^{2\alpha}\,((1-k/n)x)^{-\alpha}\,(n/k)^{\beta}\,((1-k/n)x)^{-\gamma} \\
&\quad \leqslant 4^{\gamma+2\eta}M(k/n)^{-\beta}.
\end{aligned}$$

Hence,

$$L(f_{k/n})\leqslant 4^3M(k/n)^{-\beta}.$$

Next, we compute $L(f^z)$. Let $0<x<b$, $0<z\leqslant \frac{2}{3}$, $\theta=\beta$. We have

$$\begin{aligned}
|h^{-2\alpha}x^{\beta+\alpha}\Delta_h^2 f^z(x)| &= h^{-2\alpha}x^{\beta+\alpha}\,|\Delta^2_{he_1} f(x, (1-x)z) \\
&\quad + \Delta^2_{hze_2} f(x+2h, (1-x)z) \\
&\quad + 2\Delta_{he_1}\Delta_{-hze_2} f(x+h, (1-x)z)| \\
&\leqslant 4^{2+\alpha+\gamma+2\eta}Mz^{-\gamma}.
\end{aligned}$$

Hence,

$$L(f^z)\leqslant 4^6Mz^{-\gamma}.$$

Combinig these two estimates with (6.4), we obtain

$$\begin{aligned}
&|w(x,y)(B_n(f,x,y)-f(x,y))| \\
&\quad \leqslant M_{\alpha,\gamma}x^{\beta}\sum_{k=1}^{[bn]} P_{n,k}(x)(n-k)^{-\alpha}\,(4^3M(k/n)^{-\beta}+\|f\|_\infty) \\
&\qquad + M_{\alpha,\beta}n^{-\alpha}y^{\gamma}(4^6M(y/(1-x))^{-\gamma}+\|f\|_\infty) \\
&\quad \leqslant M_{\alpha,\gamma}(1-b)^{-\alpha}\,n^{-\alpha}\left\{x^{\beta}4^3M\left(2\sum_{k=1}^{n}(n+1)/(k+1)\,P_{n,k}(x)\right)^{\beta}+\|f\|_\infty\right\} \\
&\qquad + M_{\alpha,\beta}n^{-\alpha}y^{\gamma}(4^6My^{-\gamma}+\|f\|_\infty) \\
&\quad \leqslant M'n^{-\alpha}.
\end{aligned}$$

Here M' is a constant independent of x, y, and n.

Thus, we obtain

$$\|w(x, y)(B_n(f, x, y) - f(x, y))\|_{L_\infty(x+y \leqslant 2/3)} \leqslant M'n^{-\alpha}.$$

The proof of Theorem 1 is then complete.

Remark. We may wish to extend Theorem 1 to the space $C(S)$. However, this is far more difficult and different, and we shall discuss this problem elsewhere.

Remark. By our decomposition technique here we can simplify Ditzian's proof of Theorem A (see [7]).

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